The covariant Boltzmann-Fokker-Planck equation and its associated short-time transition probability

1988 J. Phys. A: Math. Gen. 21 1017

(http://iopscience.iop.org/0305-4470/21/4/027)

J. Phys. A: Math. Gen. 21 (1988) 1017-1028. Printed in the UK

The covariant Boltzmann–Fokker–Planck equation and its associated short-time transition probability

Gerald Horwitz and Eldad Dagan

Racah Institute of Physics, The Hebrew University of Jerusalem, Jerualem, Israel

Received 29 June 1987

Abstract. In this paper we develop the short-time propagator for the Boltzmann-Fokker-Planck equation in arbitrary coordinates thereby extending our previous result which was carried out in special coordinates. This treatment uses the approach developed by Graham which uses the diffusion matrix as the metric tensor. We examine the connection between the covariant formalisms of Graham and that of Rosenbluth *et al*, showing that both formulations lead to the same form of the invariant partial differential equation for a scalar density distribution function. However, the Graham formulation has a major advantage over that of Rosenbluth *et al*, in that it is the only one which has been able to set up a functional integral for the general form of the Fokker-Planck equation. We have here extended Graham's results for the functional integral expression of the short time in general coordinates for the Boltzmann-Fokker-Planck equation and then illustrated the use of this formulation in setting up the short-time propagator for the problem of globular clusters.

1. Introduction

The main purpose of this paper is to show the way in which one introduces the path integral solution for short times consistent with the Boltzmann-Fokker-Planck equation (BFPE) in an arbitrary set of coordinates, thereby extending the results of an earlier paper. In particular we shall work out the details for an important case study of certain globular cluster models which have isotropic velocity background, in a suitable frame of reference. With the specific symmetry, the number of relevant dimensions reduces to three.

In the early 1940s Chandrasekhar had presented the Boltzmann-Fokker-Planck equation (BFPE) governing the distribution function in phase space, together with its associated short-time transition probability [1, 2]. In these works the BFPE was obtained for the case of cartesian coordinates while its corresponding short-time transition probability was introduced only the case of a diagonal diffusion matrix.

In a famous paper in 1957, Rosenbluth *et al* [3] presented a covariant formulation of the BFPE based on the general transformations of the velocity which preserve its norm. In that formulation the distribution function f_R appearing in the BFPE is identified as a scalar. In 1977 Graham [4] introduced another covariant formulation of the Fokker-Planck equation (FPE) based on the choice of the inverse of a general diffusion matrix $Q^{\mu\nu}$ as the metric tensor. In Graham's analysis the distribution f_G is a scalar density and he defines an associated scalar distribution function S for writing down the covariant form of the FPE. Graham shows the way to build up the path integral solution for such an equation [4, 5] while, together with Deininghauss [6], he derived an appropriate short-time propagator.

Rosenbluth's covariant formalism was extensively exploited for numerical calculations exploiting simplifications obtained by coordinate transformations for the BFPE [7]. However, it was not clear how to present the path integral solution for that equation. In § 2 we prove the equivalence of Rosenbluth's and Graham's formulations of the BFPE (via the introduction of a scalar density distribution function $\hat{f}_R$ associated with the scalar distribution function f_R). In that section we also extend Graham's covariant formulation to phase space.

The unification of Chandrasekhar's results, in which he presented the structure of the short-time transition probability in phase space for a specific case, with those of Graham leads in § 3 to the introduction of the short-time propagator for the BFPE in an arbitrary coordinate set, a genuine result of that paper.

In § 4 we investigate in detail a specific class of globular cluster models. We establish the appropriate BFPE for the scalar density distribution function $f_G = \hat{f}_R$ and introduce its associated short-time transition probability.

2. Covariant formulation of the BFPE

The Boltzmann equation for the change of one type of particle distribution function in cartesian phase space is given by

$$\frac{Df_R}{Dt} = \frac{\partial f_R}{\partial t} + v^\mu \frac{\partial f_R}{\partial x^\mu} + B^\mu \frac{\partial f_R}{\partial v^\mu} = \left(\frac{\partial f_R}{\partial t}\right)_C \tag{1}$$

where f_R is the number of particles per unit volume in cartesian phase space $(\boldsymbol{x}, \boldsymbol{v})$ and B^ν denotes the components of an external or self-consistent force field per unit mass. The term $(\partial f_R/\partial t)_C$ represents the change in the distribution function produced by collisions. The subscript R is introduced to differentiate between Rosenbluth's scalar distribution function f_R and Graham's scalar density distribution function f_G.

In suitable circumstances $(\partial f_R/\partial t)_C$ takes the form of the Fokker–Planck collision term:

$$\left(\frac{\partial f_R}{\partial t}\right)_C = -\frac{\partial(f_R\langle\Delta v^\mu\rangle)}{\partial v^\mu} + \frac{1}{2}\frac{\partial^2(f_R\langle\Delta v^\mu \Delta v^\nu\rangle)}{\partial v^\mu \partial v^\nu} \tag{2}$$

where $\langle\Delta v^\mu\rangle$ is the average change per unit time of the μth component of the velocity due to collisions, while $\langle\Delta v^\mu \Delta v^\nu\rangle$ is the corresponding average of mixed components. The FPE is valid if the averages of higher order may be neglected. Equation (2) may also be written as a conservation equation:

$$\left(\frac{\partial f_R}{\partial t}\right)_C = -\left(\frac{\partial f_R\langle v^\mu\rangle}{\partial v^\mu} - \frac{1}{2}\frac{\partial^2(f_R\langle\Delta v^\mu \Delta v^\nu\rangle)}{\partial v^\mu\, \partial v^\nu}\right) = -F^\mu_{R;\mu} \tag{3}$$

where $F^\mu_R = f_R\langle v^\mu\rangle - \frac{1}{2}\partial(f_R\langle\Delta v^\mu \Delta v^\nu\rangle)/\partial v^\nu$ denotes the probability current flux. The joining together of (1) and (2) leads to the BFPE:

$$\frac{\partial f_R}{\partial t} + v^\mu \frac{\partial f_R}{\partial x^\mu} + B^M \frac{\partial f_R}{\partial v^\mu} = -\frac{\partial f_R\langle v^\mu\rangle}{\partial v^\mu} + \frac{1}{2}\frac{\partial^2(f_R\langle\Delta v^\mu \Delta v^\nu\rangle)}{\partial v^\mu \partial v^\nu}. \tag{4}$$

For exploiting the symmetries of a given problem it is desirable to write the BFPE in a covariant form. Rosenbluth *et al* [3] have already presented such a framework. In their formulation the usual metric of ds_R between two adjacent points in velocity subspace is chosen as

$$(ds_R)^2 = (g_R)_{\mu\nu}\, dv^\mu\, dv^\nu. \tag{5}$$

The condition for conservation of the number of particles in a small volume element $\mathrm{d}V$ is a scalar equation $f\,\mathrm{d}V = f_{\mathrm{R}}\sqrt{g_{\mathrm{R}}}\,\mathrm{d}^3v$. Since, in the case of cartesian coordinates, $g_{\mathrm{R}} = 1$, this reduces to the standard conditions with f_{R} being seen to be a scalar. The drift K_{R}^{ν} and the diffusion matrix $Q^{\mu\nu}$ defined, respectively, as

$$K_{\mathrm{R}}^{\nu} = \langle \Delta v^{\nu} \rangle \tag{6a}$$

$$Q^{\nu\mu} = \langle \Delta v^{\nu} \Delta v^{\mu} \rangle \tag{6b}$$

are observed to transform like a contravariant vector and tensor, respectively. Under these considerations the Rosenbluth covariant form of the BFPE becomes

$$\frac{\mathrm{D}f_{\mathrm{R}}}{\mathrm{D}t} = \frac{\partial f_{\mathrm{R}}}{\partial t} + v^{\mu}\frac{\partial f_{\mathrm{R}}}{\partial x^{\mu}} + B^{\mu}\frac{\partial f_{\mathrm{R}}}{\partial v^{\mu}} = -(f_{\mathrm{R}}K_{\mathrm{R}}^{\mu})_{;\mu} + \tfrac{1}{2}(f_{\mathrm{R}}Q^{\nu\mu})_{;\nu;\mu} \tag{7}$$

where ; denotes covariant differentiation with respect to the above-mentioned metric $(g_{\mathrm{R}})_{\mu\nu}$.

We shall be interested in the covariant BFPE (7) in a slightly different form. The term $B^{\mu}\partial f_{\mathrm{R}}/\partial v^{\mu}$ may be transferred to the RHS of the equation. Then one sees that the following relation holds:

$$B^{\nu}\frac{\partial f_{\mathrm{R}}}{\partial v^{\nu}} = B^{\nu}f_{\mathrm{R},\nu} = (B^{\nu}f_{\mathrm{R}})_{;\nu} = B^{\nu}{}_{;\nu}f_{\mathrm{R}} + B^{\nu}f_{\mathrm{R};\nu}. \tag{8}$$

Since $B^{\nu}{}_{;\nu} = B^{\nu}{}_{,\nu} = 0$ in cartesian coordinates due to the independence of the force term on the velocity this, of course, holds in arbitrary coordinates. If we now define $\hat{K}_{\mathrm{R}}^{\nu}$ as a new drift term which includes the change of the velocity due to the presence of the force $\boldsymbol{B}$ as follows:

$$\hat{K}_{\mathrm{R}}^{\nu} \equiv K^{\nu} + B^{\nu} \tag{9}$$

we can write the covariant BFPE in the form:

$$\frac{\partial f_{\mathrm{R}}}{\partial t} + v^{\mu}\frac{\partial f_{\mathrm{R}}}{\partial x^{\mu}} = -(f_{\mathrm{R}}\hat{K}_{\mathrm{R}}^{\nu})_{;\nu} + \tfrac{1}{2}(f_{\mathrm{R}}Q^{\nu\mu})_{;\nu;\mu}. \tag{10}$$

This form is more suitable for introducing the short-time transition probability associated with the BFPE (cf § 3).

Let us limit ourselves for the meantime to the case where the distribution function does not depend on the spatial coordinate $\boldsymbol{x}$. For this case we have a Fokker-Planck equation with the inclusion of the force term in the drift K_{R}:

$$\partial f_{\mathrm{R}}/\partial t = -(f_{\mathrm{R}}\hat{K}_{\mathrm{R}}^{\mu})_{;\mu} + \tfrac{1}{2}(f_{\mathrm{R}}Q^{\nu\mu})_{;\nu;\mu}. \tag{11}$$

Using the explicit formulae of tensor analysis [8], equation (11) may also be written in the form:

$$\frac{\partial f_{\mathrm{R}}}{\partial t} = -\frac{1}{\sqrt{g_{\mathrm{R}}}}\frac{\partial(\sqrt{g_{\mathrm{R}}}f_{\mathrm{R}}\hat{K}_{\mathrm{R}}^{\mu})}{\partial v^{\mu}} + \frac{1}{2}\frac{1}{\sqrt{g_{\mathrm{R}}}}\frac{\partial^2(\sqrt{g_{\mathrm{R}}}f_{\mathrm{R}}Q^{\nu\mu})}{\partial v^{\mu}\,\partial v^{\nu}} + \frac{1}{2}\frac{1}{\sqrt{g_{\mathrm{R}}}}\frac{\partial(\sqrt{g_{\mathrm{R}}}(f_{\mathrm{R}})_{\mathrm{R}}\Gamma_{\gamma\nu}^{\mu}Q^{\gamma\nu})}{\partial v^{\mu}} \tag{12}$$

where $g_{\mathrm{R}} = \det(g_{\mathrm{R}})_{\nu\mu}$ and ${}_{\mathrm{R}}\Gamma_{\gamma\nu}^{\mu}$ is the Christoffel symbol of the second kind related to the metric $(g_{\mathrm{R}})_{\nu\mu}$. Multiplying both sides of (12) by $\sqrt{g_{\mathrm{R}}}$ and rearranging terms, we get the following equation:

$$\frac{\partial \hat{f}_{\mathrm{R}}}{\partial t} = -\frac{\partial((K_{\mathrm{R}}^{\mu} - \frac{1}{2}{}_{\mathrm{R}}\Gamma_{\gamma\nu}^{\mu}Q^{\gamma\nu})\hat{f}_{\mathrm{R}})}{\partial v^{\mu}} + \frac{1}{2}\frac{\partial^2(Q^{\nu\mu}\hat{f}_{\mathrm{R}})}{\partial v^{\nu}\,\partial v^{\mu}} \tag{13}$$

where

$$\hat{f}_R = \sqrt{g_R} f_R \tag{14}$$

is the scalar density distribution function associated with the scalar distribution function f_R. Equation (13) for $\hat{f}_R$ has the same form of partial differential equation in any coordinates. However, (13) is not manifestly convariant as it stands.

In 1977 Graham presented a different covariant formulation of the Fokker-Planck equation [4, 5]. He showed that the diffusion matrix $Q^{\nu\mu}(\boldsymbol{v})$ transforms like a tensor under coordinate transformations†. Since $Q^{\nu\mu}$ is a positive definite symmetric tensor, its inverse $Q_{\nu\mu}$ has the same properties and can be taken to play the role of a metric. The covariant length ds_G is defined as follows:

$$(ds_G)^2 = Q_{\nu\mu}\, dv^\nu\, dv^\mu = (g_G)_{\nu\mu}\, dv^\nu\, dv^\mu \tag{15}$$

where

$$(g_G)_{\mu\nu} = Q_{\nu\mu} \qquad Q_{\nu\mu}Q^{\mu\rho} = \delta_\nu \rho. \tag{16}$$

For a given FPE in the velocity subspace we have the equation

$$\frac{\partial f_G}{\partial t} = \frac{\partial(\hat{K}^\nu_R f_G)}{\partial v^\nu} + \frac{1}{2}\frac{\partial^2(Q^{\nu\mu} f_G)}{\partial v^\nu\, \partial v^\mu}. \tag{17}$$

Graham identifies the distribution function f_G as a scalar density (which thus differs from f_R which is a scalar). He then defined a scalar density S:

$$f_G = \sqrt{g_G} S \tag{18}$$

where $g_G = \det[(g_G)_{\nu\mu}]$. If we introduce an invariant volume element, $d\boldsymbol{\Omega}$,

$$d\boldsymbol{\Omega} = \sqrt{g_G}\, d\boldsymbol{v} \tag{19}$$

where

$$d\boldsymbol{v} = \prod_{i=1}^{3} dv^i$$

we get the covariant condition for conservation of the number of particles located in a certain volume element:

$$f_G\, d\boldsymbol{v} = S\, d\boldsymbol{\Omega}. \tag{20}$$

Graham shows that the drift term $\hat{K}^\nu_R$ does not transform like a vector and he replaces it by the covariant drift h^ν, defined as follows:

$$h^\nu = \hat{K}^\nu_R - \frac{1}{2}\frac{1}{\sqrt{g_G}}\frac{\partial(\sqrt{g_G} g^{\nu\mu}_G)}{\partial v^\mu}. \tag{21}$$

We note that in (17) and (21) we have a drift term $\hat{K}^\nu_R$ which is different from the drift term in Rosenbluth's formulation. Introducing S and h^ν into (17) and transforming ordinary derivatives into covariant ones with respect to Graham's metric, the following covariant form of the FPE is obtained:

$$S = -[h^\nu S - \tfrac{1}{2}(g^{\nu\mu}_G S)_{;\nu}]_{;\mu} \equiv -F^\nu_{G;\nu} \tag{22}$$

† Graham's exposition was based on the analysis of the relations between the Langevin equation and the FPE. The tensor properties of $Q^{\nu\mu}$ may be derived directly by the product of $\Delta v^\nu \Delta v^\mu$ of the velocity differentials in the new coordinate system expressed as a Taylor expansion of the old coordinate differentials to give $\langle \Delta v^\nu \Delta v^\mu \rangle = (\partial v^\nu/\partial v^\alpha)(\partial v^\mu/\partial v^\beta)\langle \Delta v^\alpha \Delta v^\beta \rangle + O(\Delta v^3)$.

where F_G^ν is the covariant probability current. Graham's covariant approach ensures that, in any frame of reference, the partial differential equation for the scalar density f_G retains its form as in (17). A proof of that property is given in appendix 1.

We will now show that (13) for $\hat{f}_R$ and (17) for f_G are equivalent. Let us denote by ${}_c g_G$ the value of g_G in cartesian coordinates in the velocity subspace. For a non-constant diffusion coefficient $Q^{\nu\mu}(\boldsymbol{v})$, ${}_c g_G$ is non-constant as well. On the other hand, ${}_c g_R$, the value of g_R in cartesian coordinates, is unity. If we write the transformation law for ${}_c g_G^{1/2}$ and ${}_c g_R^{1/2}$ from the cartesian coordinates v^α to the new coordinates v^β, we have the relations

$$\sqrt{g_G} = ({}_c g_G)^{1/2} |\partial v^\alpha / \partial \hat{v}^\beta| \tag{23}$$

$$\sqrt{g_R} = ({}_c g_R)^{1/2} |\partial v^\alpha / \partial \tilde{v}^\beta| = |\partial v^\alpha / \partial \tilde{v}^\beta|. \tag{24}$$

where $|\partial v^\alpha / \partial \tilde{v}^\beta|$ denotes the Jacobian of the transformation. From these equations we find that $\sqrt{g_G}$ is related to $\sqrt{g_R}$ by

$$\sqrt{g_G} = \sqrt{{}_c g_G}\,\sqrt{g_R}. \tag{25}$$

By using the fact that S is a scalar, with (18) we obtain

$$S = {}_c f_G / \sqrt{{}_c g_G} = f_G / \sqrt{g_G} \tag{26}$$

where ${}_c f_G$ denotes the Graham distribution function in cartesian coordinates. Using (25) we find

$$f_G = \sqrt{g_R}\ {}_c f_G. \tag{27}$$

By demanding that both f_R and f_G correspond to the same number of particles in a particular cartesian volume element $d\boldsymbol{v}({}_c f_G\, d\boldsymbol{v} = f_R \sqrt{g_R}\, d\boldsymbol{v})$, we get the simple connection between Graham's and Rosenbluth's distribution functions:

$$f_G = f_R \sqrt{g_R} = \hat{f}_R. \tag{28}$$

We see that f_G is just Rosenbluth's scalar distribution function multiplied by the corresponding scalar density $\sqrt{g_R}$. Looking back at (13) and (17) we find them to be identical on condition that the drift terms can be identified; the proof of this is given in appendix 2. We thus conclude that both covariant approaches based on different metrics lead to the same partial FPE for the scalar density distribution $\hat{f}_R = \sqrt{g_R} f_R = f_G$,

$$\partial \hat{f}_R / \partial t = -(\hat{f}_R K^\mu)_{,\mu} + \tfrac{1}{2}(\hat{f}_R Q^{\nu\mu})_{,\nu,\mu}. \tag{29}$$

Now we return to the spatial dependence of the BFPE. In its first version (7) we have on the LHS the total derivative Df_R/Dt. Since the term $B^\mu\, \partial f_R / \partial v^\mu$ has already been transferred into the RHS of the equation, we are left only with the term $v^\mu\, \partial f_R / \partial x^\mu$. The scalar nature of this term is retained by simply expressing the old coordinates v^μ in terms of the new ones $\tilde{v}^\mu (v^\mu = v^\mu(\tilde{v}^\nu))$. Graham did not treat the case with spatial dependence. However, using the above derived relations between Graham's and Rosenbluth's quantities, we can express the BFPE (equation (7)) in terms of S and f_G. From (18), (25) and (28), we obtain

$$S = f_R ({}_c g_G)^{-1/2}. \tag{30}$$

If we divide (29) by $({}_c g_G)^{1/2}$, we get the BFPE for S:

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial x^\mu} v^\mu(\tilde{v}^\nu) + S \frac{\partial \ln({}_c g_G)^{1/2}}{\partial x^\mu} v^\mu(\tilde{v}^\nu) = -[h^\nu - \tfrac{1}{2} g_G^{\nu\mu} S_{;\mu}]_{;\nu}. \tag{31}$$

We observe that an additional scalar term was added with respect to the original form in which S is multiplied by the spatial derivative of the Graham metric determinant in the cartesian coordinate frame. Multiplying (29) by $\sqrt{g_R}$ and using the equivalence of (17) and (13) leads to the BFPE for f_G,

$$\frac{\partial f_G}{\partial t}+v^{\mu}(\tilde{v}^{\nu})\frac{\partial f_G}{\partial x^{\mu}}=-\frac{\partial(k_G^{\nu}f_G)}{\partial v^{\nu}}+\frac{1}{2}\frac{\partial^2(g_G^{\nu\mu}f_G)}{\partial v^{\nu}\partial v^{\mu}}. \tag{32}$$

We note that in (31) a new term was added due to the possible dependence of the stochastic process on the spatial coordinates. That dependence caused the appearance of the spatial derivatives $\partial\sqrt{_c g_G}/\partial x^{\mu}$. On the other hand, $\sqrt{g_R}$ is free of such dependence and so (13) and (32) for $\hat{f}_R$ and f_G, respectively, take a similar form. Thus we have established the relationship between Rosenbluth's and Graham's covariant formulations. It is to be observed that Graham's covariant approach is adaptable to any FPE in any physical space (e.g., thermodynamic coordinates). The partial differential equation (32) was shown by Graham to be a convenient basis for defining a functional integral expression for solutions of the FPE, something which no one has been able to do in the Rosenbluth formalism. Here we are concerned with the short-time transition probability arising in Graham's approach.

3. Short-time transition probability for the BFPE

Chandrasekhar's expression for the short-time transition probability consistent with the BFPE equations (10) and (29), takes the form

$$\chi(t,\boldsymbol{x}_0,\boldsymbol{v}_0,\Delta\boldsymbol{x},\Delta\boldsymbol{v},\tau)=\Psi(t,\boldsymbol{x}_0,\boldsymbol{v}_0,\Delta\boldsymbol{v},\tau)\delta^3(\Delta\boldsymbol{x}-\boldsymbol{v}_0\tau). \tag{33}$$

$\chi(t,\boldsymbol{x}_0,\boldsymbol{v}_0,\Delta\boldsymbol{x},\Delta\boldsymbol{v},\tau)\,\mathrm{d}^3(\Delta x)\,\mathrm{d}^3(\Delta v)$ is the probability that the particle located at a prepoint $(\boldsymbol{x}_0,\boldsymbol{v}_0)$ at time t will be within phase space volume $\mathrm{d}^3(\Delta x)\,\mathrm{d}^3(\Delta v)$ around the postpoint $(\boldsymbol{x},\boldsymbol{v})$ at a later time $t+\tau$, where $\Delta\boldsymbol{x}=\boldsymbol{x}-\boldsymbol{x}_0$ and $\Delta\boldsymbol{v}=\boldsymbol{v}-\boldsymbol{v}_0$. χ has the structure of a transition probability Ψ in the velocity subspace located at the spatial point $\boldsymbol{x}_0$ multiplied by a delta function which causes the particle to advance in space just by its velocity $\boldsymbol{v}_0$. Chandrasekhar's transition probability ψ was obtained for cartesian coordinates and for a diagonal diffusion matrix only. These limitations may be removed if we use the short-time transition probability of Deininghauss and Graham (DG) instead of Chandrasekhar's ψ. That short-time transition probability, also known as the short-time propagator, is derived as the discrete expression for the path integral solution of the FPE. The explicit DG expression of the short-time propagator takes the form

$$\begin{aligned}\Psi_{DG}&(t,\boldsymbol{x}_0,\boldsymbol{v}_0,\Delta\boldsymbol{v},\tau)\\&=\Bigg[\Bigg(1+C_{\alpha\beta}(v)\eta^{\alpha}\eta^{\beta}+\frac{D_{\alpha\beta\gamma}(v)\eta^{\alpha}\eta^{\beta}\eta^{\gamma}+E_{\alpha\beta\gamma\delta}(v)\eta^{\alpha}\eta^{\beta}\eta^{\gamma}\eta^{\delta}}{\tau}\\&\quad+\frac{G_{\alpha\beta\gamma\delta\varepsilon\nu}(v)\eta^{\alpha}\eta^{\beta}\eta^{\gamma}\eta^{\delta}\eta^{\varepsilon}\eta^{\nu}}{\tau^2}\Bigg)\left(\frac{g_G}{(2\pi\tau)^3}\right)^{1/2}\exp[-\tfrac{1}{2}\tau(g_G)_{\mu\nu}(v)\\&\quad\times(\eta^{\mu}/\tau-h^{\mu}(v))(\eta^{\nu}/\tau-h^{\nu}(v))-\tfrac{1}{2}\tau(h^{\nu}{}_{;\nu}(v))+\tfrac{1}{6}R(v)]\Bigg]\end{aligned} \tag{34}$$

where $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{v}_0$ and R is the scalar curvature with the Graham metric. The various coefficients of the power series in $\boldsymbol{\eta}$ are given by the following relations:

$$C_{\alpha\beta} = -\tfrac{1}{12} R_{\alpha\beta} - \tfrac{1}{2}\{\partial(g_{.\nu} h^{\nu})/\partial v^{\cdot}\}_{\alpha\beta} \tag{35a}$$

where $R_{\alpha\beta}$ is the Ricci tensor and

$$D_{\alpha\beta\gamma} = \tfrac{1}{4}\{\partial Q_{..}/\partial v^{\cdot}\}_{\alpha\beta} \tag{35b}$$

$$E_{\alpha\beta\gamma\delta} = -\tfrac{1}{12}\{\partial^2 Q_{..}/\partial v^{\cdot}\,\partial v^{\cdot} - \tfrac{1}{12} Q_{\nu\mu} \Gamma^{\nu}_{..} \Gamma^{\mu}_{..}\}_{\alpha\beta\gamma\delta} \tag{35c}$$

$$G_{\alpha\beta\gamma\delta\varepsilon\nu} = \tfrac{1}{32}\{(\partial Q_{..}/\partial v^{\cdot})(\partial Q_{..}/\partial v^{\cdot})\}_{\alpha\beta\gamma\delta\varepsilon\nu} \tag{35d}$$

where the curly brackets denote complete symmetrisation. Substituting (34) into (33) gives the short-time transition probability consistent with the most general BFPE. In arbitrary coordinates:

$$\chi(t, \boldsymbol{x}_0, \boldsymbol{v}_0, \Delta\boldsymbol{x}, \Delta\boldsymbol{v}) = \Psi_{\mathrm{DG}}(t, \boldsymbol{x}_0, \boldsymbol{v}_0, \Delta\boldsymbol{v}, \tau)\,\delta^3(\Delta\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{v})\boldsymbol{\tau}). \tag{36}$$

This short-time transition probability actually reproduces the BFPE, equation (32). The delta function reproduces the term $v^{\mu}\partial f_{\mathrm{G}}/\partial x^{\mu}$, while ψ_{DG} reproduces the other terms in the BFPE (see § 4 of [9]). It must be emphasised that the general short-time transition probability is related to the scalar density distribution f_{G} and not to the scalar distribution f_{R}. The behaviour of f_{G} may be interpreted in terms of f_{R} by using equation (28). This fact originates because the ψ_{DG} is associated with the invariant form of the FPE as a partial differential equation and not with the covariant BFPE, equation (10), for f_{R} which does not preserve that form.

4. Globular clusters with a locally isotropic velocity distribution

We apply here the results of the previous sections in a specific case study of spherically symmetric globular star clusters composed of equal mass particles having a distribution function which depends on the velocity only through its norm. We shall introduce a specific coordinate transformation by using both Rosenbluth's and Graham's covariant formulations to reach the same BFPE. This transformation reduces the number of effective variables to three. Then we shall build the short-time transition probability for this specific BFPE equation.

The stars in this cluster are held together by their self-consistent gravitational field. A test distribution function in phase space is governed by the BFPE. The particle dynamics described by this equation is dominated by the mean self-consistent gravitational field together with the mechanism of random, weak two-body encounters with background stars (called field stars). The distribution function of the field stars at every spatial point is assumed to depend only on the magnitude of the velocity (locally isotropic background assumption). Such systems may serve as an idealisation of realistic globular clusters (the so-called Wooley and Michie-King models belong to that category) [7].

At every spatial point $\boldsymbol{r}$ we choose a set of cartesian coordinates v^i in which v^3 is directed along the radius vector $\boldsymbol{r}$ whose origin is at the centre of the cluster. In this cartesian velocity subspace, we find the following functional form for the drift:

$$\hat{K}^1_{\mathrm{R}} = \hat{K}^1_{\mathrm{G}} = \eta(v)v^1/v \qquad \hat{K}^2_{\mathrm{R}} = \hat{K}^2_{\mathrm{G}} = \eta(v)v^2/v \qquad \hat{K}^3_{\mathrm{R}} = \hat{K}^3_{\mathrm{G}} = \eta(v)v^3/v - B(r) \tag{37}$$

where $\eta(v)$ depends on the magnitude of the velocity v only and $\boldsymbol{B}(\boldsymbol{r})=-B(r)\hat{\boldsymbol{r}}$ represents the mean gravitational force. The appropriate expression for the diffusion tensor takes the form (cf § 3 of [9])

$$Q^{\nu\mu}=g_{\mathrm{G}}^{\nu\mu}=\frac{[Q_{\|}(v)-Q_{\perp}(v)]}{v^2}v^{\mu}v^{\nu}+\delta^{\nu\mu}Q_{\perp}(v) \tag{38a}$$

$$Q^{\nu\mu}=(g_{\mathrm{G}})^{\nu\mu}=\frac{[1/Q_{\|}(v)-1/Q_{\perp}(v)]}{v^2}v_{\nu}v_{\mu}+\frac{\delta^{\nu\mu}}{Q_{\perp}(v)} \tag{38b}$$

where $Q_{\|}(v)$ and $Q_{\perp}(v)$ represent the diffusion parallel and perpendicular to the particle velocity, respectively, and depend only on the magnitude of the velocity. We can also write the expression for Graham's covariant drift h^{ν}

$$h^1=h(v)v^1/v \qquad h^2=h(v)v^2/v \qquad h^3=h(v)(v^3/v)v-B(r) \tag{39}$$

where $h(v)$ is given by

$$h(v)=\eta(v)-\tfrac{1}{2}[2(Q_{\|}(v)-Q_{\perp}(v))/v+Q_{\|}(v)]+\tfrac{1}{2}(\ln g_{\mathrm{G}}^{1/2})'Q_{\|} \tag{40}$$

the prime in this equation denoting a derivative with respect to v.

Let us write down the BFPE (32) for $f_{\mathrm{R}}=f_{\mathrm{G}}$ in cartesian coordinates,

$$\frac{\partial f_{\mathrm{G}}}{\partial t}+v^3\frac{\partial f_{\mathrm{G}}}{\partial x^3}=-\frac{\partial(\hat{K}_{\mathrm{G}}^1 f_{\mathrm{G}})}{\partial v^{\nu}}+\frac{1}{2}\frac{\partial^2(g_{\mathrm{G}}^{\nu\mu}f_{\mathrm{G}})}{\partial v^{\nu}\,\partial v^{\mu}}. \tag{41}$$

Now we shall perform a coordinate transformation into new spherical coordinates $\tilde{\boldsymbol{v}}=(\boldsymbol{v},\theta,\varphi)$ at the same spatial point, where θ is the angle formed by the velocity and the radius vector $\boldsymbol{r}$ and φ is the azimuthal angle. The various relations between the two coordinate frames are given by

$$v^1=v\sin\theta\cos\varphi \qquad v^2=v\sin\theta\sin\varphi \qquad v^3=v\cos\theta. \tag{42}$$

From these transformations we obtain the following expressions for the partial derivatives of the new coordinates in terms of the old ones and vice versa:

$$\frac{\partial v^{\alpha}}{\partial\tilde{v}^{\beta}}=\begin{bmatrix}\sin\theta\cos\varphi & v\cos\theta\cos\varphi & -v\sin\theta\sin\varphi\\ \sin\theta\sin\varphi & v\cos\theta\sin\varphi & v\sin\theta\cos\varphi\\ \cos\theta & -v\sin\theta & 0\end{bmatrix} \tag{43a}$$

$$\frac{\partial\tilde{v}^{\mu}}{\partial v^{\nu}}=\begin{bmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\\ \cos\theta\cos\varphi/v & \cos\theta\sin\varphi/v & -\sin\theta/v\\ \sin\varphi/v\sin\theta & \cos\varphi/v\sin\theta & 0\end{bmatrix}. \tag{43b}$$

These relations enable us to transform the diffusion tensor explicitly to obtain its diagonal form in the new coordinate system:

$$\tilde{Q}^{vv}={}_{\mathrm{G}}\tilde{g}^{vv}=Q_{\|}(v) \qquad \tilde{Q}^{\theta\theta}={}_{\mathrm{G}}\tilde{g}^{\theta\theta}=Q_{\perp}(v)/v^2 \qquad \tilde{Q}^{\varphi\varphi}={}_{\mathrm{G}}\tilde{g}^{\varphi\varphi}=Q_{\perp}(v)/v^2\sin^2\theta. \tag{44}$$

The inverse of the diffusion tensor is found immediately to take the form

$${}_{\mathrm{G}}\tilde{g}_{vv}=(Q_{\|}(v))^{-1} \qquad {}_{\mathrm{G}}\tilde{g}_{\theta\theta}=v^2/Q_{\perp} \qquad {}_{\mathrm{G}}\tilde{g}_{\varphi\varphi}=v^2\sin^2\theta/Q_{\perp} \tag{45}$$

and we also have the following expression for $\sqrt{g_{\mathrm{G}}}$:

$$\sqrt{g_{\mathrm{G}}}=\frac{v^2\sin\theta}{Q_{\perp}\sqrt{Q_{\|}}}. \tag{46}$$

h^ν transforms like a vector, whereby we find its explicit form in the new coordinates to be

$$\tilde{h}^v = h(v) = B\cos\theta \qquad \tilde{h}^\theta = B\sin\theta/v \qquad \tilde{h}^\varphi = 0. \tag{47}$$

Using (21) and (40) to eliminate $\tilde{K}^\mu_{\mathrm{G}}$ we finally obtain the BFPE for f_{G} in the new coordinates

$$\begin{aligned}\frac{\partial \tilde{f}_{\mathrm{G}}}{\partial t} + v\cos\theta\frac{\partial \tilde{f}_{\mathrm{G}}}{\partial x^3} - \left\{\left(\eta(v) - B\cos\theta + \frac{Q_\perp}{v}\right)\tilde{f}_{\mathrm{G}}\right\}_{,v} \\ - \left\{-\left(\frac{B\sin\theta}{v} + \frac{1}{2}\frac{Q_\perp}{v^2\cot\theta}\right)\tilde{f}_{\mathrm{G}}\right\}_{,\theta} + \tfrac{1}{2}(Q_\parallel \tilde{f}_{\mathrm{G}})_{,v,v} + \frac{1}{2}\left(\frac{Q_\perp}{v^2}\tilde{f}_{\mathrm{G}}\right)_{,\theta,\theta} \\ + \frac{1}{2}\left\{\left(\frac{Q_\perp}{v^2\sin^2\theta}\right)\tilde{f}_{\mathrm{G}}\right\}_{,\varphi,\varphi}.\end{aligned} \tag{48}$$

In the case where f_{G} does not depend on the angle φ, we can omit the last term on the RHS of (48). We observe that the spherical symmetry in the problem has reduced the BFPE to two dimensions in the velocity subspace and one spatial coordinate.

We proceed to carry out the same transformation according to Rosenbluth's covariant approach. To do that we need the explicit expressions for the Chrisoffel symbols ${}_{\mathrm{R}}\Gamma^{\cdot}_{\cdot\cdot}$ as follows:

$$\begin{array}{lll} {}_{\mathrm{R}}\tilde{\Gamma}^1_{22} = -v & {}_{\mathrm{R}}\tilde{\Gamma}^2_{12} = {}_{\mathrm{R}}\tilde{\Gamma}^2_{12} = v^{-1} & {}_{\mathrm{R}}\tilde{\Gamma}^3_{13} = {}_{\mathrm{R}}\tilde{\Gamma}^3_{21} = v^{-1} \\ \tilde{\Gamma}^1_{33} = -v\sin^2\theta & \tilde{\Gamma}^2_{33} = -\sin\theta\cos\theta & {}_{\mathrm{R}}\tilde{\Gamma}^3_{23} = {}_{\mathrm{R}}\tilde{\Gamma}^3_{32} = \cot\theta. \end{array} \tag{49}$$

By transforming K^ν_{R} as a vector, we find that the Rosenbluth drift term of (13) is identical to the Graham drift term $\tilde{K}^\nu_{\mathrm{G}}$ and so we get the same BFPE, equation (48), for $f_{\mathrm{G}} = \hat{f}_{\mathrm{R}}$.

Finally, we can introduce the short-time propagator for (48). According to (36), the short-time propagator for the BFPE in spherical coordinates is given by

$$\chi = \Psi_{\mathrm{DG}}\delta(\Delta x^3 - v\cos\theta). \tag{50}$$

Ψ_{DG} can be expressed by (34) and (35) with the introduction of the Christoffel symbols in the Graham metric, ${}_{\mathrm{G}}\Gamma^{\cdot}_{\cdot\cdot}$. Since the diffusion tensor is diagonal, the computation is much simplified and gives the following expressions for the non-vanishing Christoffel symbols:

$$\begin{array}{lll} {}_{\mathrm{G}}\Gamma^v_{vv} = -\tfrac{1}{2}(\ln Q_\parallel)' & {}_{\mathrm{G}}\Gamma^v_{\theta\theta} = -\tfrac{1}{2}Q_\parallel(v^2/Q_\perp)' & {}_{\mathrm{G}}\Gamma^v_{\theta\theta} = -Qv\sin\theta \\ {}_{\mathrm{G}}\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta & {}_{\mathrm{G}}\Gamma^\varphi_{\varphi v} = {}_{\mathrm{G}}\Gamma^\theta_{\varphi v} = v^{-1} - \tfrac{1}{2}(\ln Q_\parallel)' & {}_{\mathrm{G}}\Gamma^\varphi_{\varphi\theta} = \cot\theta. \end{array} \tag{51}$$

5. Summary and discussion

Having shown that both the Rosenbluth and Graham covariant formulations are associated with the same form-invariant BFPE for the scalar density distribution function $f_{\mathrm{R}} = f_{\mathrm{G}}$, we have used the relations based on that equivalence to extend Graham's covariant formulation to the BFPE in phase space. Graham's BFPE was found to have the appropriate structure to introduce a functional integral expression for the short-time transition probability associated with that equation in arbitrary coordinates.

In addition to its utility in defining a functional integral, Graham's covariant formulation has the additional advantage of a wider range of application than that of Rosenbluth, since it can be applied to any FPE in arbitrary coordinates (e.g. thermodynamic variables).

The introduction of the short-time transition probability may have various advantages. Besides the theoretical aspects connected with the propagator for a given equation one can also use numerical methods unique to the path integral approach [10] (see also the summary of [9]).

The short-time transition probability is composed of two parts: the δ function which relates to the deterministic behaviour in space, while the transition probability in the velocity subspace represents the stochastic behaviour. The same structure of the propagator is also found in a paper by Grabert *et al* [11] which deals with macroscopic thermodynamic variables. The dynamics of the variables connected with reversible fluxes is described by the δ function (singular diffusion matrix) while the variables connected with fluctuations are described by means of the FPE and its associated (Graham) covariant propagator.

We have finally applied this formalism to the evolution of globular star clusters. For quasisteady evolutionary stages of globular clusters one finds that the dynamical time for completing one revolution in a certain orbit is much larger than the timescale typical for velocity changes due to small-angle two-body collisions. In this case, one averages the collisional effects over an orbit to get an orbit-averaged two-dimensional FPE with the variables (E, J) where E and J denote the energy and the norm of the angular momentum of a star in an orbit. In this equation Rosenbluth's metric loses its meaning while the use of Graham's metric for covariant coordinate transformations is still valid. Such an additional transformation is desirable since under certain conditions one can find a new coordinate system in which the diffusion matrix becomes a constant [12]. Calculations done in a coordinate system for which we have a constant diffusion coefficient are much simplified.

Acknowledgments

We would like to thank H Dekker, R Graham and J Katz for useful comments.

Appendix 1. The FPE for S and f_G compared

By writing the covariant differential $F^{\nu}_{G;\nu}$ in terms of $g_G^{1/2}$, we get the equation

$$\dot{S} = -g_G^{-1/2}(g_G^{1/2} F_G^{\nu})_{,\nu} = -g_G^{-1/2}[g_G^{1/2}(h^{\nu} - \tfrac{1}{2} g_G^{\mu\nu} S_{;\mu})]_{,\nu} \tag{A1.1}$$

where the comma denotes ordinary differentiation. Multiplying both sides of (A1.1) by $g_G^{1/2}$ and using (18) and (21), we obtain

$$\begin{aligned} g_G^{1/2}\dot{S} = \dot{f}_G &= \left\{ K_G^{\nu} f_G - \tfrac{1}{2}\frac{\partial g_G}{\partial v^{\mu}} f_G - \tfrac{1}{2} g_G^{\nu\mu} \frac{\partial \ln(g_G)^{1/2}}{\partial v^{\mu}} - \tfrac{1}{2} g_G^{\nu\mu}\frac{\partial f_G}{\partial v^{\mu}} - \tfrac{1}{2} g_G^{\nu\mu} f_G \frac{\partial \ln(g_G)^{-1/2}}{\partial v^{\mu}} \right\}_{,\nu} \\ &= [K_G^{\nu} f_G - \tfrac{1}{2}(Q^{\nu\mu} f_G)_{,\mu}]_{,\nu} \end{aligned} \tag{A1.2}$$

which is identical to (17).

Appendix 2. Equivalence of the drift terms in the two formalisms

Let us take the FPE in cartesian coordinates as a starting point for both Graham's and Rosenbluth's formalisms and use the label c for that frame of reference to write the FPE as

$$\frac{\partial_{\mathrm{c}} f}{\partial t}=-\frac{\partial({}_{\mathrm{c}}\hat{K}^{\nu}_{\mathrm{G}}\,{}_{\mathrm{c}}f)}{\partial v^{\nu}}-\frac{1}{2}\frac{\partial^2({}_{\mathrm{c}}Q^{\nu\mu}\,{}_{\mathrm{c}}f)}{\partial v^{\nu}\,\partial v^{\mu}}. \tag{A2.1}$$

In these coordinates the Rosenbluth drift term ${}_{\mathrm{c}}\hat{K}^{\nu}_{\mathrm{R}}$ coincides with the Graham drift term $\hat{K}^{\nu}_{\mathrm{G}}$. We observe that the values of both distribution functions f_{R} and f_{G} are the same according to (28) (${}_{\mathrm{c}}f_{\mathrm{R}}={}_{\mathrm{c}}f_{\mathrm{G}}={}_{\mathrm{c}}f$). Now we shall perform a coordinate transformation into $\tilde{v}^{\alpha}$. If we use (21) to express $\hat{K}^{\nu}_{\mathrm{G}}$ in the new frame and write the explicit transformation of ${}_{\mathrm{c}}h^{\nu}$ as a vector, we obtain

$$\begin{aligned}\tilde{\hat{K}}^{\nu}_{\mathrm{G}}=\tilde{h}^{\nu}+\tfrac{1}{2}(1/\sqrt{\tilde{g}_{\mathrm{R}}})\frac{\partial((\tilde{g}_{\mathrm{G}})^{1/2}\tilde{g}^{\nu\mu}_{\mathrm{G}})}{\partial\tilde{v}^{\mu}}&=\left({}_{\mathrm{c}}\hat{K}^{\alpha}-\tfrac{1}{2}\frac{\partial_{\mathrm{c}}g^{\alpha\mu}_{\mathrm{G}}}{\partial v^{\mu}}-\tfrac{1}{2}{}_{\mathrm{c}}g^{\alpha\mu}_{\mathrm{G}}\frac{\partial(\ln({}_{\mathrm{c}}g_{\mathrm{G}})^{1/2})}{\partial v^{\mu}}\right)\frac{\partial\tilde{v}^{\nu}}{\partial v^{\alpha}}\\&\quad+\tfrac{1}{2}\frac{\partial\tilde{g}^{\nu\mu}_{\mathrm{G}}}{\partial\tilde{v}^{\nu}}+\tfrac{1}{2}\tilde{g}^{\nu\mu}_{\mathrm{G}}\left(\frac{\partial(\ln({}_{\mathrm{c}}g_{\mathrm{G}})^{1/2})}{\partial\tilde{v}^{\mu}}+\frac{\partial\ln{}_{\mathrm{c}}g^{1/2}_{\mathrm{R}}}{\partial\tilde{v}^{\mu}}\right)\end{aligned} \tag{A2.2}$$

where we have also used (25). We shall add and subtract the term $\frac{1}{2}{}_{\mathrm{R}}\tilde{\Gamma}^{\nu}_{\nu\mu}g^{\omega\mu}_{\mathrm{G}}$ on the RHS of (A2.2), where ${}_{\mathrm{R}}\tilde{\Gamma}^{\nu}_{\omega\mu}$ is the Christoffel symbol associated with the Rosenbluth metric in the new coordinates. So we get

$$\begin{aligned}\tilde{K}^{\nu}_{\mathrm{G}}&={}_{\mathrm{c}}K^{\alpha}\partial\tilde{v}^{\nu}/\partial v^{\alpha}-\tfrac{1}{2}{}_{\mathrm{R}}\tilde{\Gamma}^{\nu}_{\omega\mu}\tilde{Q}^{\omega\mu}\\&\quad-\frac{1}{2}\left[-\frac{\partial_{\mathrm{c}}g^{\alpha\mu}_{\mathrm{G}}}{\partial\tilde{v}^{\mu}}\frac{\partial\tilde{v}^{\nu}}{\partial v^{\alpha}}+\left(\frac{\partial\tilde{g}^{\nu\mu}_{\mathrm{G}}}{\partial\tilde{v}^{\mu}}+\tilde{g}^{\nu\mu}_{\mathrm{G}}\frac{\partial(\ln g^{1/2}_{\mathrm{R}})}{\partial\tilde{v}^{\mu}}+{}_{\mathrm{R}}\tilde{\Gamma}^{\nu}_{\omega\mu}\tilde{g}^{\omega\mu}_{\mathrm{G}}\right)\right]\\&\quad+\left(-{}_{\mathrm{c}}g^{\alpha\mu}_{\mathrm{G}}\frac{\partial\ln{}_{\mathrm{c}}g^{1/2}_{\mathrm{G}}}{\partial v^{\mu}}\frac{\partial\tilde{v}^{\nu}}{\partial v^{\alpha}}+\tilde{g}^{\nu\mu}_{\mathrm{G}}\frac{\partial\ln{}_{\mathrm{c}}g^{1/2}_{\mathrm{G}}}{\partial\tilde{v}^{\mu}}\right).\end{aligned} \tag{A2.3}$$

The first two terms on the RHS are identical with the Rosenbluth drift terms of (13), since $\hat{K}^{\nu}={}_{\mathrm{c}}\hat{K}^{\alpha}\partial\tilde{v}^{\nu}/\partial v^{\alpha}$ transforms like a vector and also due to the equality $\tilde{g}^{\nu\mu}_{\mathrm{G}}=\tilde{Q}^{\nu\mu}$. The middle term cancels because we can identify it as a subtraction of the transformed Rosenbluth covariant vector ${}_{\mathrm{c}}g^{\alpha\mu}_{\mathrm{G};\mu}=g^{\alpha\mu}_{\mathrm{G},\mu}$ with its equivalent expression $\tilde{g}^{\nu\mu}_{\mathrm{G};\mu}$. In the last term we identify a subtraction of the transformed covariant vector ${}_{\mathrm{c}}g^{\alpha\mu}_{\mathrm{G}}\,\partial\ln({}_{\mathrm{c}}g_{\mathrm{G}})^{1/2}/\partial v^{\mu}$ with its equivalent expression $\tilde{g}^{\alpha\mu}_{\mathrm{G}}(\partial\ln({}_{\mathrm{c}}g_{\mathrm{G}})^{1/2}/\partial\tilde{v}^{\mu})$. (Note that $(\ln{}_{\mathrm{c}}g^{1/2}_{\mathrm{G}})_{;\mu}={}_{\mathrm{c}}g^{1/2}_{\mathrm{G},\mu}$ is a lower index vector because it is formed by derivation of the scalar $\ln{}_{\mathrm{c}}g^{1/2}_{\mathrm{G}}$.)

References

[1] Chandrasekhar S 1943 *Rev. Mod. Phys.* **15** 1-89

[2] Chandrasekhar S 1960 *Principles of Stellar Dynamics* (New York: Dover)

[3] Rosenbluth M N, MacDonald W M and Judd D L 1957 *Phys. Rev.* **107** 1-67

[4] Graham R 1977 *Z. Phys.* B **26** 281-90

[5] Graham R 1977 *Z. Phys.* B **26** 397-405

[6] Deininghauss U and Graham R 1969 *Z. Phys.* B **34** 211-9

[7] Ipser J L 1977 *Astrophys. J.* **218** 846-56
Lightman A P and Shapiro S L 1978 *Rev. Mod. Phys.* **50** 437-81
Henon C M 1973 *Dynamical Structure and Evolution of Stellar Systems* (*Saas Fee Course*) ed L Martinet and F Mayor (Genève: Observatoire de Genève) pp 183-260
Cohn H and Kulsrud R A 1978 *Astrophys. J.* **226** 1087
Cohn H 1979 *Astrophys. J.* **234** 1036-53; 1980 *Astrophys. J.* **242** 765-71

[8] Synge J L and Schild A 1949 *Tensor Calculus* (Toronto: Toronto University Press)
Papapetrou A 1974 *Lectures on General Relativity* (Dordrecht: Reidel)
[9] Dagan E and Horwitz G 1988 *J. Phys. A: Math. Gen.* **21** 1001-15
[10] Wehner M F and Wolfer W G 1983 *Phys. Rev.* A **27** 2663-70, **28** 3003-11
[11] Grabert H, Grabert R G and Green M S 1980 *Phys. Rev.* A **21** 2136-46
[12] Garrido L, Lurie D and San Miguel M 1979 *J. Stat. Phys.* **21** 313-35